PEARSON

SPECIALIST MATHEMATICS 12

QUEENSLAND

STUDENT BOOK

UNITS 3 & 4

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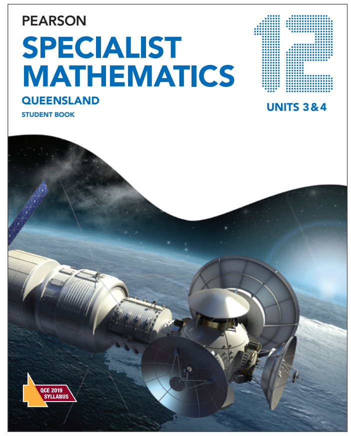

Contents

Supporting the integrating of technology

Students are supported with the integration of technology in a number of ways. The eBook includes 'How to' user guides covering all basic functionality for the following three graphing calculators:

- TI-84 Plus CE
- TI-Nspire CX (non-CAS)
- CASIO fx-CG50AU

Throughout the student book you will find Technology worked examples strategically placed within the theory for both the TI-Nspire CX (non-CAS) and CASIO fx-CG50AU. The examples clearly demonstrate how the technology can be used effectively and efficiently for the content being covered in that chapter.

Graphing calculators are not the only technology integrated throughout the Pearson Queensland senior mathematics series. Spreadsheets, Desmos and interactive widgets have been included to provide students with the opportunity to visualise concepts, consolidate their understanding and make mathematical connections.

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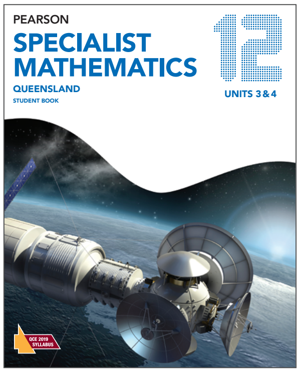


Specialist Mathematics 12
Student book

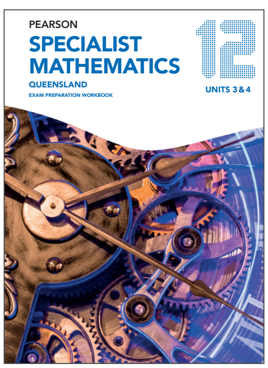
Student book

The student book has been authored by local authors, ensuring quality content and complete curriculum coverage for Queensland, enabling students to prepare with ease and confidence. We have covered the breadth of the content within our exercise questions, from simpler skills-focused questions to those using unfamiliar contexts and application of the theory learnt. The theory, worked examples and question sets are written in line with the assessment objectives, with the aim of familiarising students with QCE cognitive verbs in the process of dependent and guided instruction. Additional interactives that help explain the theory and consolidate concepts have been included throughout all chapters.

Pearson Reader+

Pearson Reader+ is our next-generation eBook. This is an electronic textbook that students can access on any device, online or offline. It is linked to features, interactives and visual media that will help consolidate students' understanding of concepts and ideas, as well as other useful content specifically developed for senior mathematics. It supports students with appropriate online resources and tools for every section of the student book, providing access to exemplar worked solutions that demonstrate high levels of mathematical and everyday communication. Students will have the opportunity to learn independently through the Explore further tasks and Making connections interactive widgets, which have been designed to engage and support conceptual understanding. Additionally, teachers have access to syllabus maps, a teaching program, sample exams, problem-solving and modelling tasks, and additional banks of questions for extra revision.

Specialist Mathematics 12
eBook



Specialist Mathematics 12
Exam preparation workbook

Exam preparation workbook

Additional component for Year 12 only

The *Exam preparation workbook* provides additional support in preparing students for the external exam. It has been constructed to guide the students through a sequence of preparatory steps and build confidence leading up to the external exam.

How to use this book

Pearson Specialist Mathematics 12 Queensland Units 3 & 4

This Queensland senior mathematics series has been written by a team of experienced Queensland teachers for the QCE 2019 syllabus. It offers complete curriculum coverage, rich content and comprehensive teacher support.

(d) Calculate the maximum height reached by the ball during its flight. Give your answer correct to 1 decimal place.

1 The ball reaches its maximum height when the $\hat{j}$ component of $\dot{r}$ is zero (i.e. the vertical component of velocity is zero).	$20 - 9.8t = 0$ $t = \frac{20}{9.8}$ $= 2.04\text{ s}\,(2\text{ d.p.})$
2 Its height is given by the $\hat{j}$ component of $\boldsymbol{r}$.	$h_{max} = h(2.04)$ $= 2.04\left(20 - 4.9 \times \frac{20}{9.8}\right)$ $= 2.04 \times 10$ $= 20.4\text{ m}\,(1\text{ d.p.})$
3 Interpret the answer.	The ball reaches a maximum height of 20.4 m at 2.04 s.

Explore further — Projectile motion: Explore the effects of initial velocity, both speed and angle, on a projectile.

Making connections — Investigating projectile motion: Adjust time value to trace locus of projectile. Position, acceleration and velocity vectors have been drawn in.

Technology worked example — Investigating projectile motion: Understand how the particle moves while changing the vector direction.

EXERCISE 3.4 Investigating curvilinear motion

Worked Example 18

1 A particle moves so that its position vector $\boldsymbol{r}$ at time t is given by $\boldsymbol{r} = 3\cos(2t)\hat{i} + 3\sin(2t)\hat{j}, t \geq 0$.
(a) Demonstrate that the particle moves in a circle and determine the Cartesian equation of its path.
(b) Demonstrate that the particle moves with constant speed.
(c) Demonstrate that the particle's acceleration has constant magnitude and is perpendicular to the direction of motion of the particle.

2 The position vector of a particle at time t seconds, $t \geq 0$, is $\boldsymbol{r} = (1 + \sin(4t))\hat{i} + (2 - \cos(4t))\hat{j}$ (units are metres).
(a) Determine the Cartesian equation of the path of the particle and sketch the result.
(b) Prove that its acceleration is always perpendicular to its velocity.

3 The position vector of a particle at time t is given by $\boldsymbol{r} = \sin(2t)\hat{i} + \cos(2t)\hat{j} + \sin(2t)\hat{k}$.
(a) Which of the following will give the correct equation for the acceleration vector?
A $\boldsymbol{a} = -\boldsymbol{r}$ B $\boldsymbol{a} = -2\boldsymbol{r}$ C $\boldsymbol{a} = -4\boldsymbol{r}$ D $\boldsymbol{a} = 4\boldsymbol{v}$
(b) Explain the common error made by a student who selected the first incorrect option.

Explore further

This eBook feature provides an opportunity for students to consolidate their understanding of concepts and ideas with the aid of technology, and answer a small number of questions to deepen their understanding and broaden their skills base. These activities should take approximately 5–15 minutes to complete.

Making connections

This eBook feature provides teachers and students with a visual interactive of specific mathematics concepts or ideas to aid students in their understanding.

Tech-free questions

These questions are designed to provide students with the opportunity to practice algebraic manipulations to prepare them for technology-free examination papers.

Worked solutions

Fully worked solutions are provided for every question in the student textbook and can be accessed from the accompanying eBook.

Highlighting common errors

Throughout the exercises, authors have integrated questions designed to highlight common errors frequently made by students. Explanations are given in the worked solutions.

Technology worked examples

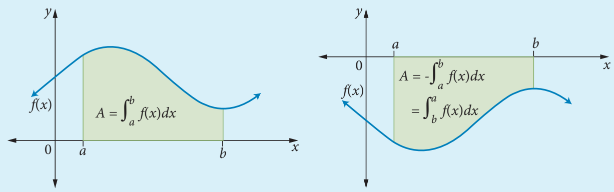
These worked examples offer support in using technology such as spreadsheets, graphing calculators and graphing software, and include technology-focused worked examples and activities.

Key information

Key information and rules are highlighted throughout the chapter.

The area bounded by a function $f(x)$, the x-axis, and the lines $x = a$ and $x = b$ (where $a < b$) is equal to:

$\int_a^b f(x)\,dx$ if the function lies above the x-axis from a to b, and $-\int_a^b f(x)\,dx$ if the function lies below the x-axis from a to b.



Every worked example and question is graded

Every example and question is graded using the three levels of difficulty, as specified in the QCE syllabus:

- simple familiar (1 bar)
- complex familiar (2 bars)
- complex unfamiliar (3 bars)

The visibility of this grading helps ensure all levels of difficulty are well covered.

4 Using the chain rule to solve a related rates problem

A spherical hot air balloon develops a leak so that t minutes later the radius r metres is given by $r(t) = 24 - 3t^2$.

Assuming the balloon remains spherical (that is, $V(r) = \frac{4\pi}{3}r^3$), determine the rate at which the balloon is losing air 2 minutes after the leak commenced.

THINKING	WORKING
1 Identify the key elements in the problem then write the given information and the required rate.	Given: $V(r) = \frac{4\pi}{3}r^3$ $r(t) = 24 - 3t^2$ Required to determine: $\left.\frac{dV}{dt}\right\|_{t=2}$
2 Apply the chain rule to write the required rate of change in terms of other related rates.	Volume is an explicit parametric function of time, so $V(t) \equiv V(r(t))$ $\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$
3 Substitute the derivatives needed into the related rates equation.	$\frac{dV}{dt} = \frac{d}{dr}\left(\frac{4\pi}{3}r^3\right) \times \frac{d}{dt}\left(24 - 3t^2\right)$ $= 4\pi\left(r^2\right) \times (-6t)$ $= 4\pi\left(24 - 3t^2\right)^2 \times (-6t)$
4 Calculate the required rate at the given time.	$\left.\frac{dV}{dt}\right\|_{t=2} = -6912\pi$
5 Interpret the solution in the context of the question and write the answer.	After 2 minutes, the balloon is losing air at a rate of 6912π cubic metres per minute.

WARNING

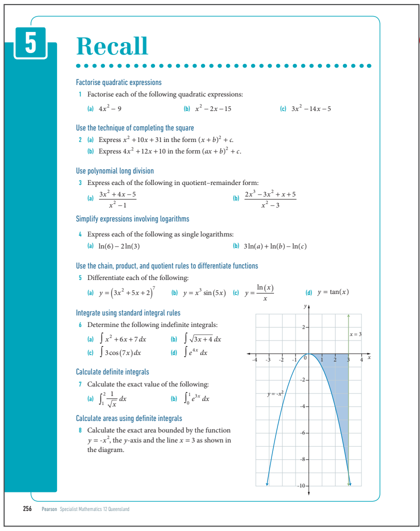
Never substitute values for changing quantities into a related rates problem until after all derivatives have been taken. Substituting too soon makes changeable variables behave like constants, with zero derivatives.

Meeting the needs of the QCE syllabus

The authors have integrated both the cognitive verbs and the language of the syllabus objectives throughout the worked examples and questions.

Warning boxes

Warning boxes are located throughout the chapter to alert students to common errors and misconceptions.



Recall

Factorise quadratic expressions

Use the technique of completing the square

Use polynomial long division

Simplify expressions involving logarithms

Use the chain, product, and quotient rules to differentiate functions

Integrate using standard integral rules

Calculate definite integrals

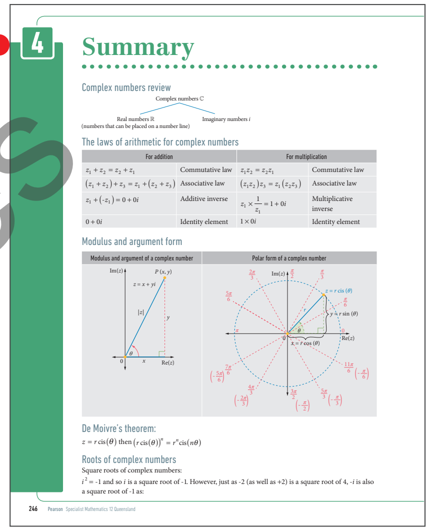
Calculate areas using definite integrals

Recall

Each chapter begins with a review of assumed knowledge for the chapter.

Summary

At the end of each chapter, there is a summary of the key facts and rules discussed in the chapter.



Summary

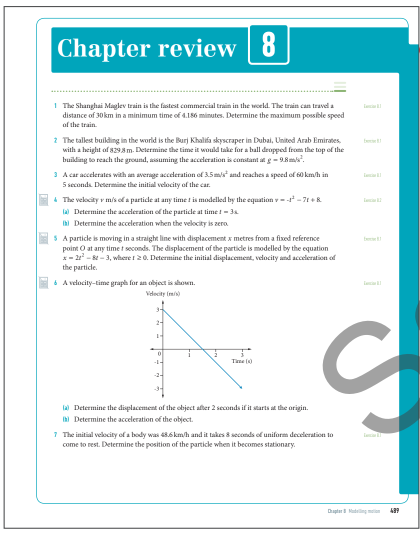
Complex numbers review

The laws of arithmetic for complex numbers

Modulus and argument form

De Moivre's theorem:

Roots of complex numbers



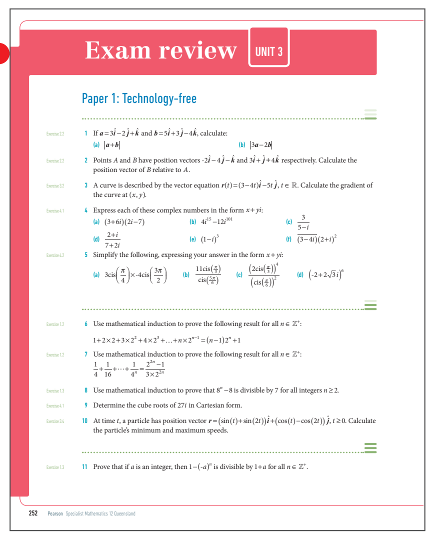
Chapter review 8

Chapter review

Every chapter review follows the QCAA examination proportions for level of difficulty, which is 60% simple familiar, 20% complex familiar and 20% complex unfamiliar.

Exam review

Exam reviews provide cumulative practice of content already covered, to prepare students for the end-of-year exam. They have been placed at the end of each Unit.



Exam review UNIT 3

Paper 1: Technology-free

3

Vector calculus

3

Recall

Differentiate

1 Calculate $\frac{dx}{dt}$ for each of the following:

(a) $x = 1 - \cos(t)$ (b) $x = \cos^2(t)$ (c) $x = e^{2t} - 4t^3$ (d) $x = \sec(t)$

2 Calculate $\frac{dy}{dt}$ for each of the following:

(a) $y = 4\sin(t)$ (b) $y = \sin^2(t)$ (c) $y = \log_e(2t)$ (d) $y = \tan(t)$

Calculate the derivative for a given value

3 If $f(t) = 10t - 5t^2$, calculate $f'(2)$.

4 If $f(t) = \sin(2t)$, calculate without using technology $f'\left(\frac{\pi}{6}\right)$.

5 If $f(t) = 3\cos^2(\pi t)$, calculate without using technology $f'\left(\frac{1}{4}\right)$.

6 If $f(t) = \sqrt{2t+1}$, calculate $f'\left(\frac{1}{2}\right)$.

Anti-differentiate

7 Calculate the following:

(a) $\int\left(e^{2t} + e^{-t}\right)dt$ (b) $\int 3\sin(2t)\,dt$ (c) $\int(4t+1)^2\,dt$ (d) $\int\frac{1}{3t+2}\,dt$

Solve problems involving displacement and velocity

8 A particle moves in a straight line such that its position, x metres, relative to an origin O at time t seconds is given by $x(t) = -t^2 + 2t - 8$, $t \geq 0$.

(a) State the particle's initial position. (b) Determine the particle's velocity at $t = 4$.

(c) Determine when the particle's velocity is zero.

9 The velocity, v m/s, of a particle moving in a straight line at time t seconds is given by $v(t) = 6 - 2t$. At time $t = 0$, the particle is 3 metres to the right of the origin O.

(a) Determine $x(t)$. (b) Describe the particle's position relative to O after 4 seconds.

(c) Calculate the total distance travelled by the particle after 4 seconds.

Calculate the magnitude of a vector

10 Calculate the magnitude of the following vectors:

(a) $\boldsymbol{a} = 2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}$ (b) $\boldsymbol{c} = -5\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} - 2\sqrt{2}\hat{\boldsymbol{k}}$

Use the scalar product

11 Let $\boldsymbol{a} = \hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$, $\boldsymbol{b} = -3\hat{\boldsymbol{i}} - 5\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$ and $\boldsymbol{c} = 2\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} - 4\hat{\boldsymbol{k}}$. Calculate:

(a) $\boldsymbol{a} \cdot \boldsymbol{b}$ (b) $\boldsymbol{a} \cdot \boldsymbol{c}$ (c) $\boldsymbol{b} \cdot \boldsymbol{c}$

12 Calculate the angle, in radians, between each of the following pairs of vectors. State each angle correct to three significant figures.

(a) $\boldsymbol{a} = \hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$ and $\boldsymbol{b} = 2\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}}$ (b) $\boldsymbol{a} = 3\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} - 6\hat{\boldsymbol{k}}$ and $\boldsymbol{b} = -\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + 5\hat{\boldsymbol{k}}$

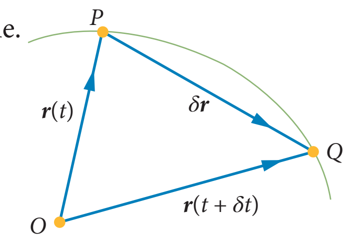
13 Let $\boldsymbol{a} = -\hat{\boldsymbol{i}} + m\hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}}$ and $\boldsymbol{b} = m\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} - 4\hat{\boldsymbol{k}}$, $m \in \mathbb{R}$. If $\boldsymbol{a}$ is perpendicular to $\boldsymbol{b}$, determine the value of m.

3.2 Differentiation of vector functions

Consider a function f that is differentiable throughout its domain.

The derivative of f is defined by $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

By analogy, you can define the derivative of a vector function of a real variable.



Suppose $\boldsymbol{r}(t)$ and $\boldsymbol{r}(t+\delta t)$ are position vectors of neighbouring points P and Q on the smooth continuous curve defined by $\boldsymbol{r}(t)$.

By the triangle law of vector addition:

$$\overrightarrow{OP} + \overrightarrow{PQ} = \overrightarrow{OQ}$$

$$\overrightarrow{PQ} = \delta \boldsymbol{r}$$

$$\begin{aligned} \delta \boldsymbol{r} &= \overrightarrow{PQ} \\ &= \overrightarrow{OQ} - \overrightarrow{OP} \\ &= \boldsymbol{r}(t+\delta t) - \boldsymbol{r}(t) \end{aligned}$$

So $\frac{\delta \boldsymbol{r}}{\delta t} = \frac{\boldsymbol{r}(t+\delta t) - r(t)}{\delta t}$, $\delta t \neq 0$.

Provided that the limit exists, the derivative $\frac{d\boldsymbol{r}}{dt}$ is defined as:

$$\begin{aligned} \frac{d\boldsymbol{r}}{dt} &= \lim_{\delta t \to 0} \frac{\delta \boldsymbol{r}}{\delta t} \\ &= \lim_{\delta t \to 0} \frac{\boldsymbol{r}(t+\delta t) - \boldsymbol{r}(t)}{\delta t} \end{aligned}$$

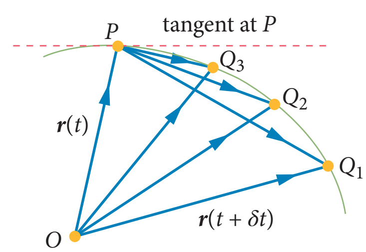
Re-expressing $\frac{\delta \boldsymbol{r}}{\delta t}$ as $\frac{1}{\delta t}(\delta \boldsymbol{r})$, see that $\frac{\delta \boldsymbol{r}}{\delta t}$ is a vector parallel to $\delta \boldsymbol{r}$. i.e. $\frac{\delta \boldsymbol{r}}{\delta t}$ is a vector whose direction is along $\overrightarrow{PQ}$.

By analogy with real (scalar) calculus, $\frac{d\boldsymbol{r}}{dt}$ is denoted as the rate of change of $\boldsymbol{r}$ with respect to t at the point P.

As $\delta t \to 0$, Q approaches P (shown by $Q_1, Q_2, Q_3 \ldots$).

As $\delta t \to 0$, $\boldsymbol{r}(t+\delta t) \to \boldsymbol{r}(t)$ and so the direction of $\frac{\delta \boldsymbol{r}}{\delta t}$ successively approaches the tangent at P.

Therefore, a geometric interpretation for $\frac{d\boldsymbol{r}}{dt}$ has been established.



Rules for differentiation of vector functions

Vector functions can be differentiated in a similar manner to real functions.

Derivative of a constant vector

If $\boldsymbol{r} = \boldsymbol{c}$, where $\boldsymbol{c}$ is a constant vector, then $\frac{d\boldsymbol{r}}{dt} = 0$.

This result can be verified as follows:

$$\begin{aligned} \frac{d\boldsymbol{r}}{dt} &= \lim_{\delta t \to 0} \frac{\delta \boldsymbol{r}}{\delta t} \\ &= \lim_{\delta t \to 0} \frac{\boldsymbol{r}(t+\delta t) - \boldsymbol{r}(t)}{\delta t} \\ &= \lim_{\delta t \to 0} \frac{\boldsymbol{c} - \boldsymbol{c}}{\delta t} \\ &= \lim_{\delta t \to 0} \frac{\boldsymbol{0}}{\delta t} \\ &= \boldsymbol{0} \end{aligned}$$

Note that $\frac{d\hat{\boldsymbol{i}}}{dt} = \frac{d\hat{\boldsymbol{j}}}{dt} = \frac{d\hat{\boldsymbol{k}}}{dt} = \boldsymbol{0}$.

Derivative of a product of a scalar function and a constant vector c

If $\boldsymbol{r} = s\boldsymbol{c}$, where s is a function of t and $\boldsymbol{c}$ is a constant vector, then $\frac{d\boldsymbol{r}}{dt} = \frac{ds}{dt}\boldsymbol{c}$.

This result can be verified as follows:

$$\begin{aligned}\frac{d\boldsymbol{r}}{dt} &= \lim_{\delta t \to 0} \frac{\delta \boldsymbol{r}}{\delta t} \\ &= \lim_{\delta t \to 0} \frac{\boldsymbol{r}(t+\delta t) - \boldsymbol{r}(t)}{\delta t} \\ &= \lim_{\delta t \to 0} \frac{s(t+\delta t)\boldsymbol{c} - s(t)\boldsymbol{c}}{\delta t} \\ &= \lim_{\delta t \to 0} \frac{\left[s(t+\delta t) - s(t)\right]\boldsymbol{c}}{\delta t} \\ &= \frac{ds}{dt}\boldsymbol{c}\end{aligned}$$

Derivative of a sum of vectors

If $\boldsymbol{u}$ and $\boldsymbol{v}$ are functions of t, and if $\boldsymbol{r} = \boldsymbol{u} + \boldsymbol{v}$, then $\frac{d\boldsymbol{r}}{dt} = \frac{d\boldsymbol{u}}{dt} + \frac{d\boldsymbol{v}}{dt}$.

This result can be verified as follows:

$$\begin{aligned}\frac{d\boldsymbol{r}}{dt} &= \lim_{\delta t \to 0} \frac{\delta \boldsymbol{r}}{\delta t} \\ &= \lim_{\delta t \to 0} \frac{\boldsymbol{r}(t+\delta t) - \boldsymbol{r}(t)}{\delta t} \\ &= \lim_{\delta t \to 0} \frac{\boldsymbol{u}(t+\delta t) + \boldsymbol{v}(t+\delta t) - \left[\boldsymbol{u}(t) + \boldsymbol{v}(t)\right]}{\delta t} \\ &= \lim_{\delta t \to 0} \frac{\left[\boldsymbol{u}(t+\delta t) - \boldsymbol{u}(t)\right] + \left[\boldsymbol{v}(t+\delta t) - \boldsymbol{v}(t)\right]}{\delta t} \\ &= \frac{d\boldsymbol{u}}{dt} + \frac{d\boldsymbol{v}}{dt}\end{aligned}$$

Using the last two rules and the fact that $\hat{\boldsymbol{i}}$, $\hat{\boldsymbol{j}}$ and $\hat{\boldsymbol{k}}$ are constant vectors, gives:

If $\boldsymbol{r} = x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}} + z\hat{\boldsymbol{k}}$, then $\frac{d\boldsymbol{r}}{dt} = \frac{dx}{dt}\hat{\boldsymbol{i}} + \frac{dy}{dt}\hat{\boldsymbol{j}} + \frac{dz}{dt}\hat{\boldsymbol{k}}$ and $\frac{d^2\boldsymbol{r}}{dt^2} = \frac{d^2x}{dt^2}\hat{\boldsymbol{i}} + \frac{d^2y}{dt^2}\hat{\boldsymbol{j}} + \frac{d^2z}{dt^2}\hat{\boldsymbol{k}}$.

If $\boldsymbol{r}(t) = x(t)\hat{\boldsymbol{i}} + y(t)\hat{\boldsymbol{j}} + z(t)\hat{\boldsymbol{k}}$, then $\boldsymbol{r}'(t) = x'(t)\hat{\boldsymbol{i}} + y'(t)\hat{\boldsymbol{j}} + z'(t)\hat{\boldsymbol{k}}$ and $\boldsymbol{r}''(t) = x''(t)\hat{\boldsymbol{i}} + y''(t)\hat{\boldsymbol{j}} + z''(t)\hat{\boldsymbol{k}}$.

Hence, a vector expressed in component form can be differentiated term by term.

8 Differentiating a vector function

Determine $\frac{d\boldsymbol{r}}{dt}$ and $\frac{d^2\boldsymbol{r}}{dt^2}$ if $\boldsymbol{r} = 3t^2\hat{\boldsymbol{i}} + \left(4t - t^3\right)\hat{\boldsymbol{j}} + \sin(5t)\hat{\boldsymbol{k}}$.

THINKING	WORKING
1 Differentiate each component of $\boldsymbol{r}$ to determine $\frac{d\boldsymbol{r}}{dt}$.	$\boldsymbol{r} = 3t^2\hat{\boldsymbol{i}} + \left(4t - t^3\right)\hat{\boldsymbol{j}} + \sin(5t)\hat{\boldsymbol{k}}$ $\frac{d\boldsymbol{r}}{dt} = 6t\hat{\boldsymbol{i}} + \left(4 - 3t^2\right)\hat{\boldsymbol{j}} + 5\cos(5t)\hat{\boldsymbol{k}}$

2 Differentiate each component of $\frac{d\boldsymbol{r}}{dt}$ to determine $\frac{d^2\boldsymbol{r}}{dt^2}$.	$\frac{d\boldsymbol{r}}{dt} = 6t\hat{\boldsymbol{i}} + \left(4 - 3t^2\right)\hat{\boldsymbol{j}} + 5\cos(5t)\hat{\boldsymbol{k}}$ $\frac{d^2\boldsymbol{r}}{dt^2} = 6\hat{\boldsymbol{i}} - 6t\hat{\boldsymbol{j}} - 25\sin(5t)\hat{\boldsymbol{k}}$

9 Differentiating and substituting a value into a vector function

Determine the value of $\boldsymbol{r}'(0)$ and $\boldsymbol{r}''(0)$ if $\boldsymbol{r}(t) = \sin(t)\hat{\boldsymbol{i}} + 2t^2\hat{\boldsymbol{j}} + e^{-2t}\hat{\boldsymbol{k}}$.

THINKING	WORKING
1 Differentiate each component of $\boldsymbol{r}(t)$ to determine $\boldsymbol{r}'(t)$.	$\boldsymbol{r}(t) = \sin(t)\hat{\boldsymbol{i}} + 2t^2\hat{\boldsymbol{j}} + e^{-2t}\hat{\boldsymbol{k}}$ $\boldsymbol{r}'(t) = \cos(t)\hat{\boldsymbol{i}} + 4t\hat{\boldsymbol{j}} - 2e^{-2t}\hat{\boldsymbol{k}}$
2 Substitute $t = 0$ into $\boldsymbol{r}'(t)$.	$\boldsymbol{r}'(0) = \cos(0)\hat{\boldsymbol{i}} + 4(0)\hat{\boldsymbol{j}} - 2e^{-2(0)}\hat{\boldsymbol{k}}$ $= \hat{\boldsymbol{i}} - 2\hat{\boldsymbol{k}}$
3 Differentiate each component of $\boldsymbol{r}'(t)$ to determine $\boldsymbol{r}''(t)$.	$\boldsymbol{r}'(t) = \cos(t)\hat{\boldsymbol{i}} + 4t\hat{\boldsymbol{j}} - 2e^{-2t}\hat{\boldsymbol{k}}$ $\boldsymbol{r}''(t) = -\sin(t)\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} + 4e^{-2t}\hat{\boldsymbol{k}}$
4 Substitute $t = 0$ into $\boldsymbol{r}''(t)$.	$\boldsymbol{r}''(0) = -\sin(0)\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} + 4e^{-2(0)}\hat{\boldsymbol{k}}$ $= 4\hat{\boldsymbol{j}} + 4\hat{\boldsymbol{k}}$
5 Write the answer.	$\boldsymbol{r}'(0) = \hat{\boldsymbol{i}} - 2\hat{\boldsymbol{k}}$ $\boldsymbol{r}''(0) = 4\hat{\boldsymbol{j}} + 4\hat{\boldsymbol{k}}$

Consider a curve described by the vector equation $\boldsymbol{r}(t) = x(t)\hat{\boldsymbol{i}} + y(t)\hat{\boldsymbol{j}}$. The gradient of the curve at the point (x, y) can be found using related rates, namely:

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx}, \text{ where } \frac{dt}{dx} = \frac{1}{\frac{dx}{dt}}.$$

Worked example 10 illustrates this.

10 Calculating the gradient at a point on a curve defined by a vector function

A curve is described by the vector equation $\boldsymbol{r}(t) = 2t\hat{\boldsymbol{i}} + (t+2)\hat{\boldsymbol{j}}, t \geq 0$.

(a) Calculate the gradient of the curve at the point when $t = 2$.

THINKING	WORKING
1 Write and number the parametric equations.	$x = 2t$ [1] $y = t + 2$ [2]

2 Calculate $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

$\frac{dx}{dt} = 2$ and $\frac{dy}{dt} = 1$

3 Use the chain rule $\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx}$ to determine $\frac{dy}{dx}$ where $t = 2$.

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx}$$
$$= 1 \times \frac{1}{2}$$
$$= \frac{1}{2}$$

4 Interpret the answer.

The gradient of the curve at $t = 2$ is $\frac{1}{2}$.

(b) Calculate $\boldsymbol{r}'(t)$.

Differentiate each component of $\boldsymbol{r}(t)$ to determine $\boldsymbol{r}'(t)$.

$$\boldsymbol{r}(t) = 2t\hat{\boldsymbol{i}} + (t+2)\hat{\boldsymbol{j}}$$
$$\boldsymbol{r}'(t) = 2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}},\ t \geq 0$$

(c) Determine a unit vector, $\hat{\boldsymbol{s}}$, parallel to the tangent to the curve at the point when $t = 2$.

1 Use $\hat{\boldsymbol{s}} = \frac{\boldsymbol{s}}{|\boldsymbol{s}|}$.

$\boldsymbol{s} = \boldsymbol{r}'(t)$ then $|\boldsymbol{s}| = \sqrt{2^2 + 1^2} = \sqrt{5}$

2 Write the answer.

$$\hat{\boldsymbol{s}} = \frac{\boldsymbol{s}}{|\boldsymbol{s}|}$$
$$= \frac{(2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}})}{\sqrt{5}}$$
$$= \frac{1}{\sqrt{5}}(2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}})$$

EXERCISE 3.2 Differentiation of vector functions

Worked Example

1 Calculate $\frac{d\boldsymbol{r}}{dt}$ and $\frac{d^2\boldsymbol{r}}{dt^2}$ for each of the following vector functions: 8

(a) $\boldsymbol{r} = 5t^2\hat{\boldsymbol{i}} - 2t^3\hat{\boldsymbol{j}} - t\hat{\boldsymbol{k}}$

(b) $\boldsymbol{r} = 4\sin(2t)\hat{\boldsymbol{i}} + 3\cos(2t)\hat{\boldsymbol{j}} + t^2\hat{\boldsymbol{k}}$

2 Calculate $\boldsymbol{r}'(\alpha)$ and $\boldsymbol{r}''(\alpha)$ for each of the following vector functions and values of α: 9

(a) $\boldsymbol{r}(t) = -\sin(2t)\hat{\boldsymbol{i}} + \cos(2t)\hat{\boldsymbol{j}} + e^{-t}\hat{\boldsymbol{k}},\ \alpha = 0$

(b) $\boldsymbol{r}(t) = e^{-t}\hat{\boldsymbol{i}} + e^{t}\hat{\boldsymbol{j}} + (t^3 - 1)\hat{\boldsymbol{k}},\ \alpha = 0$

(c) $\boldsymbol{r}(t) = 10t\hat{\boldsymbol{i}} + (10\sqrt{3} - 5t^2)\hat{\boldsymbol{j}},\ \alpha = 2$

(d) $\boldsymbol{r}(t) = \sec(t)\hat{\boldsymbol{i}} + \tan(t)\hat{\boldsymbol{j}},\ \alpha = \frac{\pi}{4}$

Worked Example 10

3 A curve is described by the vector equation $\boldsymbol{r}(t) = t\hat{\boldsymbol{i}} + \frac{1}{t}\hat{\boldsymbol{j}}, t > 0$.

(a) Calculate the gradient of the curve at the point when $t = 1$.

(b) Calculate $\boldsymbol{r}'(t)$.

(c) Determine a unit vector, $\hat{\boldsymbol{s}}$, parallel to the tangent to the curve where $t = 1$.

4 A curve is described by the vector equation $\boldsymbol{r}(t) = (t^2 + t)\hat{\boldsymbol{i}} + (t^2 - t)\hat{\boldsymbol{j}}, t \geq 0$.

(a) Calculate the gradient of the curve at the point when $t = 0$.

(b) Calculate $\boldsymbol{r}'(t)$.

(c) Determine a unit vector, $\hat{\boldsymbol{s}}$, parallel to the tangent to the curve where $t = 0$.

5 A curve is described by the vector equation $\boldsymbol{r}(t) = \cos(2t)\hat{\boldsymbol{i}} + \cos(t)\hat{\boldsymbol{j}}, 0 \leq t \leq 2\pi$.

(a) Calculate the gradient of the curve at the point when $t = \frac{\pi}{3}$.

(b) Calculate $\boldsymbol{r}'(t)$.

(c) Determine a unit vector, $\hat{\boldsymbol{s}}$, parallel to the tangent to the curve where $t = \frac{\pi}{3}$.

6 For each of the following vector functions, calculate $\frac{d\boldsymbol{r}}{dt}$ and determine any restrictions on t.

(a) $\boldsymbol{r} = t\left(\sin(t)\hat{\boldsymbol{i}} + \cos(t)\hat{\boldsymbol{j}}\right)$

(b) $\boldsymbol{r} = t\left(\log_e(t)\hat{\boldsymbol{i}} + e^t\hat{\boldsymbol{j}}\right)$

7 A curve is described by the vector equation $\boldsymbol{r}(t) = (5t + 2)\hat{\boldsymbol{i}} + 3t\hat{\boldsymbol{j}}, t \in \mathbb{R}$. The gradient of the curve at (x, y) is:

A 3

B 5

C $\frac{3}{5}$

D $\frac{5}{3}$

8 A curve is described by the vector equation $\boldsymbol{r}(t) = \cos^3(t)\hat{\boldsymbol{i}} + \sin^3(t)\hat{\boldsymbol{j}}, 0 \leq t \leq 2\pi$.

(a) Calculate the gradient of the curve at the point when $t = \frac{2\pi}{3}$.

(b) Calculate $\boldsymbol{r}'(t)$.

(c) Determine a unit vector, $\hat{\boldsymbol{s}}$, parallel to the tangent to the curve where $t = \frac{2\pi}{3}$.

9 A curve is described by the vector equation $\boldsymbol{r}(t) = e^{2t}\hat{\boldsymbol{i}} + \sin(2t)\hat{\boldsymbol{j}}, t \geq 0$. Calculate the corresponding time t at which the gradient of this curve is equal to $\frac{1}{2}$.

10 A curve C is defined by the parametric equations $x = 1 + t$ and $y = t^2, t \geq 0$.

(a) Determine the Cartesian equation of C and determine its domain.

(b) Sketch the graph of C.

(c) Determine the vector equation of C.

(d) Determine a unit vector, $\hat{\boldsymbol{s}}$, parallel to the tangent to the curve where $t = 1$.

11 The position vector of a particle at time t, is given by $\boldsymbol{r}(t) = \sin(t)\hat{\boldsymbol{i}} + t\hat{\boldsymbol{j}} + \cos(t)\hat{\boldsymbol{k}}$. Prove that $\boldsymbol{r}'(t)$ is perpendicular to $\boldsymbol{r}''(t)$.

12 The position vector of a particle at time t is given by $\boldsymbol{r}(t) = 4t\hat{\boldsymbol{i}} + 4t^2\hat{\boldsymbol{j}}, t \geq 0$. Determine when the magnitude of the angle between $\boldsymbol{r}'(t)$ and $\boldsymbol{r}''(t)$ is $\frac{\pi}{4}$.

Velocity and acceleration vectors

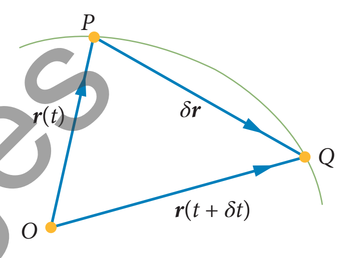
3.3

You have learned how a moving particle can be defined by a position vector, in terms of time, t. You have also learned how it is possible to differentiate a vector function.

In your study of instantaneous rates of change, you will be aware of the relationship between the process of differentiation and the calculation of velocity and acceleration for rectilinear motion. Vectors offer a useful method for extending these calculations to objects moving in three dimensions.

Velocity vector

Consider again the vector $\boldsymbol{r}(t)$, which represents the position of a particle at point P at a certain time, t. After a short time, δt, the particle is at Q with position vector $\boldsymbol{r}(t+\delta t)$.



The displacement (or change in position) of the particle during the time interval δt is:

$$\delta\boldsymbol{r} = \overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \boldsymbol{r}(t+\delta t) - \boldsymbol{r}(t)$$

Since velocity is the time ratio of change of displacement, the *average velocity* of the particle during this time interval is:

$$\frac{\delta\boldsymbol{r}}{\delta t} = \frac{\boldsymbol{r}(t+\delta t)-\boldsymbol{r}(t)}{\delta t}, \delta t \neq 0$$

Re-expressing $\frac{\delta\boldsymbol{r}}{\delta t}$ as $\frac{1}{\delta t}(\delta\boldsymbol{r})$, you will see that $\frac{\delta\boldsymbol{r}}{\delta t}$ is a vector parallel to $\delta\boldsymbol{r}$. i.e. $\frac{\delta\boldsymbol{r}}{\delta t}$ is a vector whose direction is along $\overrightarrow{PQ}$.

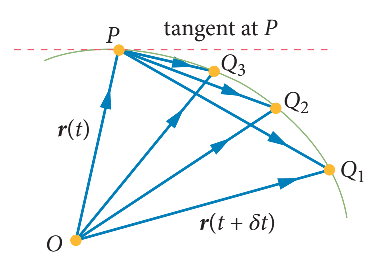
The instantaneous velocity of P at time t is therefore given by $\lim\limits_{\delta t \to 0} \frac{\boldsymbol{r}(t+\delta t)-\boldsymbol{r}(t)}{\delta t}$.

By analogy with scalar calculus, this rate of change limit is denoted by $\frac{d\boldsymbol{r}}{dt}$ or $\dot{\boldsymbol{r}}$.

If $\boldsymbol{r}(t)$ defines the position vector of a particle P at time t, then the velocity at P is:

$$\boldsymbol{v} = \boldsymbol{r}'(t) = \dot{\boldsymbol{r}} = \frac{d\boldsymbol{r}}{dt} = \lim_{\delta t \to 0} \frac{\boldsymbol{r}(t+\delta t)-\boldsymbol{r}(t)}{\delta t}$$

The direction of $\boldsymbol{v}$ is the direction of $\overrightarrow{PQ}$ as $\delta t \to 0$. i.e. as $Q \to P$ (see following diagram).



Therefore, the instantaneous velocity of the particle, $\boldsymbol{v}$, is parallel to the path of the particle at P, in the direction of motion.

If $\boldsymbol{v} = \frac{d\boldsymbol{r}}{dt}$, i.e. the velocity of the particle is the rate of change of position, then $v = |\boldsymbol{v}|$ is the speed of the particle.

11 Calculating the velocity vector

A particle moves so that its position vector at time t is given by $\boldsymbol{r}(t) = t\hat{\boldsymbol{i}} + (t-2)^2\,\hat{\boldsymbol{j}}, t \geq 0$.

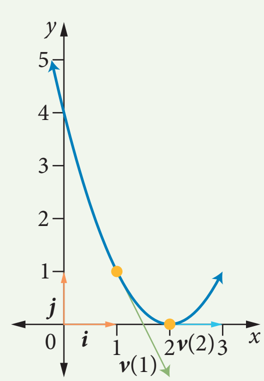
(a) Determine an expression for the velocity of the particle at time t.

THINKING	WORKING
Differentiate each component with respect to t to calculate $\boldsymbol{v}(t)$.	$\boldsymbol{r}(t) = t\hat{\boldsymbol{i}} + (t-2)^2\,\hat{\boldsymbol{j}}$ $\boldsymbol{v} = \dfrac{d\boldsymbol{r}}{dt}$ $\boldsymbol{v}(t) = \hat{\boldsymbol{i}} + (2t-4)\,\hat{\boldsymbol{j}}$

(b) Calculate the velocity of the particle at $t = 1$ and $t = 2$.

Substitute the given values of t into $\boldsymbol{v}(t)$.	$\boldsymbol{v}(t) = \hat{\boldsymbol{i}} + (2t-4)\,\hat{\boldsymbol{j}}$ $\boldsymbol{v}(1) = \hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}$ $\boldsymbol{v}(2) = \hat{\boldsymbol{i}}$

(c) Sketch the path of the particle and draw the velocity vectors at $t = 1$ and $t = 2$.

1 Write and number the parametric equations.	$x = t$ [1] $y = (t-2)^2$ [2]
2 Write the Cartesian equation with domain.	The Cartesian equation is $y = (x-2)^2, x \geq 0$.
3 Determine the location of the particle at $t = 1$ and $t = 2$. These will determine the location for the tails of the velocity vectors.	At $t = 1$: $\boldsymbol{r}(1) = 1\hat{\boldsymbol{i}} + (1-2)^2\,\hat{\boldsymbol{j}}$ $= 1\hat{\boldsymbol{i}} + 1\hat{\boldsymbol{j}}$ At $t = 2$: $\boldsymbol{r}(1) = 2\hat{\boldsymbol{i}} + (2-2)^2\,\hat{\boldsymbol{j}}$ $= 2\hat{\boldsymbol{i}} + 0\hat{\boldsymbol{j}}$
4 Sketch the parabola with correct domain and velocity vectors at $t = 1$ and $t = 2$.	 The velocity vectors are tangential to the particle's path at the respective points.

Acceleration vector

Acceleration is the rate of change of velocity with respect to time. Calculation of acceleration from the velocity vector is somewhat similar to the calculation of velocity from the position vector, as previously described.

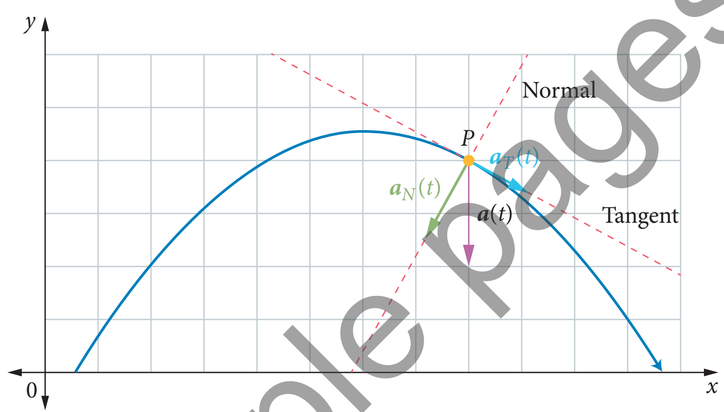
Unless the particle is moving in a straight line, $\boldsymbol{a}$ is *not* parallel to $\boldsymbol{v}$.

$$\boldsymbol{a} = \boldsymbol{v}'(t) = \dot{\boldsymbol{v}} = \frac{d\boldsymbol{v}}{dt} = \lim_{\delta t \to 0} \frac{\boldsymbol{v}(t+\delta t) - \boldsymbol{v}(t)}{\delta t}$$

or, equivalently,

$$\boldsymbol{a} = \boldsymbol{r}''(t) = \ddot{\boldsymbol{r}} = \frac{d^2\boldsymbol{r}}{dt^2}$$

In general, $\boldsymbol{a}$ has a component *tangential* to the curve at P (the direction of $\boldsymbol{v}$) and a component *normal* to the curve at P. To understand how this works, consider the case of *projectile motion* (a form of motion you will study in greater depth later in this chapter). In simplified projectile motion, the only acceleration acting on a particle is due to gravity (and this acceleration acts in a downwards direction). The path traced out by the point is parabolic, and the acceleration vector $\boldsymbol{a}(t)$ can be split into the normal vector $\boldsymbol{a}_N(t)$, shown in green, and the tangential vector $\boldsymbol{a}_T(t)$, shown in blue:



Consider also a scenario where a car is travelling around a circular track at constant speed. Is the car accelerating?

By definition, the car is accelerating if its velocity is changing. Although the magnitude of the car's velocity (its speed) is constant, the car is continually changing direction so that its velocity is changing. Hence, the car is accelerating.

Making connections

Observe the vector direction and acceleration of a particle

Move the sliders to observe how the vector acceleration changes.

12 Calculating the acceleration vector and resolving it into tangential and normal components

A particle moves so that its position vector at time t is given by $\boldsymbol{r}(t) = t\hat{\boldsymbol{i}} + (t-2)^2\,\hat{\boldsymbol{j}}$, $t \geq 0$.

(a) Determine an expression for the acceleration of the particle at time t.

THINKING	WORKING
1 Differentiate each component with respect to time to determine $\boldsymbol{v}(t)$.	$\boldsymbol{r}(t) = t\hat{\boldsymbol{i}} + (t-2)^2\,\hat{\boldsymbol{j}}$ $\boldsymbol{v}(t) = \boldsymbol{r}'(t) = \hat{\boldsymbol{i}} + (2t-4)\hat{\boldsymbol{j}}$

2 Differentiate again to determine $\boldsymbol{a}(t)$.

$\boldsymbol{v}(t) = \hat{\boldsymbol{i}} + (2t - 4)\hat{\boldsymbol{j}}$

$\boldsymbol{a} = \dfrac{d\boldsymbol{v}}{dt}$

$\boldsymbol{a}(t) = 2\hat{\boldsymbol{j}}$

The particle has a constant acceleration throughout its motion.

(b) Resolve the acceleration vector at $t = 1$ into tangential and normal components, showing them to scale on a diagram.

1 Calculate a unit vector parallel to $\boldsymbol{v}(1)$.

$\boldsymbol{v}(1) = \hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}$ and so a unit vector parallel to it is $\hat{\boldsymbol{v}}(1) = \dfrac{1}{\sqrt{5}}\left(\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}\right)$

2 Calculate the vector projection of $\boldsymbol{a}$ onto $\hat{\boldsymbol{v}}(1)$ to determine the tangential component of $\boldsymbol{a}$.

$$\begin{aligned}(\boldsymbol{a} \cdot \hat{\boldsymbol{v}}(1))\hat{\boldsymbol{v}}(1) &= \left(2\hat{\boldsymbol{j}} \cdot \frac{1}{\sqrt{5}}\left(\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}\right)\right)\frac{1}{\sqrt{5}}\left(\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}\right)\\ &= \left(-\frac{4}{\sqrt{5}}\right)\left(\frac{1}{\sqrt{5}}\right)(\boldsymbol{i} - 2\boldsymbol{j})\\ &= -\frac{4}{5}\left(\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}\right)\end{aligned}$$

3 Calculate the vector projection of $\boldsymbol{a}$ perpendicular to $\hat{\boldsymbol{v}}(1)$ to determine the normal component of $\boldsymbol{a}$.

The normal component of $\boldsymbol{a}$ is given by $\boldsymbol{a} - (\boldsymbol{a} \cdot \hat{\boldsymbol{v}}(1))\hat{\boldsymbol{v}}(1)$.

$$\begin{aligned}\boldsymbol{a} - (\boldsymbol{a} \cdot \hat{\boldsymbol{v}}(1))\hat{\boldsymbol{v}}(1) &= 2\hat{\boldsymbol{j}} - \left(-\frac{4}{5}\left(\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}\right)\right)\\ &= 2\hat{\boldsymbol{j}} + \frac{4}{5}\left(\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}\right)\\ &= \frac{4}{5}\hat{\boldsymbol{i}} + \frac{2}{5}\hat{\boldsymbol{j}}\\ &= \frac{2}{5}\left(2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}\right)\end{aligned}$$

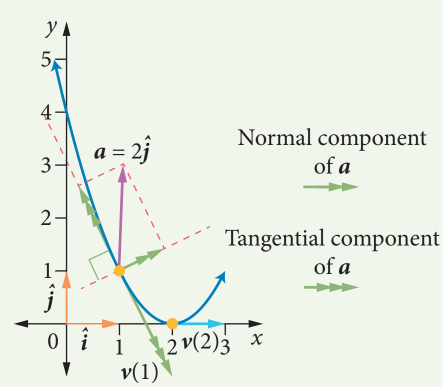
4 Determine the position of the particle at $t = 1$. This will be the point from which the acceleration vectors will be sketched.

$\boldsymbol{r}(t) = t\hat{\boldsymbol{i}} + (t - 2)^2\hat{\boldsymbol{j}}$

$\boldsymbol{r}(1) = 1\hat{\boldsymbol{i}} + 1\hat{\boldsymbol{j}}$

Therefore, draw the vectors from the point $(1, 1)$.

5 Sketch the tangential and normal components of $\boldsymbol{a}$ to scale.



(c) Calculate the angle between the velocity and the acceleration vectors at $t = 1$. Give your answer in degrees correct to 1 decimal place.

1 Determine the relevant vectors.	$\boldsymbol{v}(1) = \hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}$ $\boldsymbol{a}(1) = 2\hat{\boldsymbol{j}}$
2 Substitute into the dot product formula $\boldsymbol{a} \cdot \boldsymbol{b} = \lvert\boldsymbol{a}\rvert\lvert\boldsymbol{b}\rvert\cos(\theta)$.	Let θ be the angle between $\boldsymbol{v}(1)$ and $\boldsymbol{a}$, where $\boldsymbol{v}(1) \cdot \boldsymbol{a} = \lvert\boldsymbol{v}(1)\rvert\lvert\boldsymbol{a}\rvert\cos(\theta)$. $\left(\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}\right) \cdot 2\hat{\boldsymbol{j}} = \left(\sqrt{5}\right)(2)\cos(\theta)$
3 Rearrange the equation to make $\cos(\theta)$ the subject.	$-4 = \left(\sqrt{5}\right)(2)\cos(\theta)$ $\cos(\theta) = -\frac{2}{\sqrt{5}}$ $\theta = \cos^{-1}\left(-\frac{2}{\sqrt{5}}\right)$
4 Express θ in degrees correct to 1 decimal place.	$\theta = 153.4°$ (1 d.p.)
5 Interpret the answer.	The angle between the velocity and acceleration vectors at $t = 1$ is 153.4°.

Technology worked example
Tangential and normal components of an acceleration vector

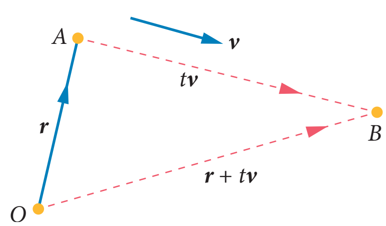
Calculations involving position, velocity and acceleration vectors

In the following figure, a particle is at A with position vector $\boldsymbol{r}$ at time $t = 0$.

If the particle is moving with *constant* velocity $\boldsymbol{v}$, then it is moving in a straight line and in time interval t it undergoes a displacement $t\boldsymbol{v}$.

Hence at time t it has position vector $\overrightarrow{OB} = \boldsymbol{r} + t\boldsymbol{v}$.

The following worked example uses this result.



13 Particles moving with a constant velocity vector

Two passenger ferries, F and G, travelling at constant velocities are observed from a clifftop at 11:00 am. Their position and velocity vectors are:

$\boldsymbol{r}_F = 30\hat{\boldsymbol{i}} - 15\hat{\boldsymbol{j}}$ $\quad\boldsymbol{v}_F = -10\hat{\boldsymbol{i}} + 15\hat{\boldsymbol{j}}$

$\boldsymbol{r}_G = -10\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}}$ $\quad\boldsymbol{v}_G = 10\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}}$

where distance is measured in kilometres and time in hours.

Determine whether the ferries will collide if they maintain their velocities. If there is a collision, determine when the collision will occur. Identify any assumptions and comment on the effects of these assumptions on your solution.

THINKING	WORKING
1 Use $\boldsymbol{r} + t\boldsymbol{v}$ to determine the position vectors of both ferries at time t hours after 11:00 am.	Ferry F: $\boldsymbol{r}_F = 30\hat{\boldsymbol{i}} - 15\hat{\boldsymbol{j}} + t\left(-10\hat{\boldsymbol{i}} + 15\hat{\boldsymbol{j}}\right)$ $= (30 - 10t)\hat{\boldsymbol{i}} + (15t - 15)\hat{\boldsymbol{j}}$ Ferry G: $\boldsymbol{r}_G = -10\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}} + t(10\hat{\boldsymbol{i}} + 15\hat{\boldsymbol{j}})$ $= (10t - 10)\hat{\boldsymbol{i}} + (15t + 5)\hat{\boldsymbol{j}}$
2 The two ferries will collide if there is a value of t (>0) such that $\boldsymbol{r}_F = \boldsymbol{r}_G$.	$(30 - 10t)\hat{\boldsymbol{i}} + (15t - 15)\hat{\boldsymbol{j}} = (10t - 10)\hat{\boldsymbol{i}} + (15t + 5)\hat{\boldsymbol{j}}$
3 Equate the $\hat{\boldsymbol{i}}$ components and solve for t.	$30 - 10t = 10t - 10$ $20t = 40$ $t = 2$
4 Equate the $\hat{\boldsymbol{j}}$ components and solve for t.	$15t - 15 = 5t + 5$ $10t = 20$ $t = 2$
5 Write a conclusion.	$\boldsymbol{r}_F(2) = \boldsymbol{r}_G(2)$, hence the ferries will collide when $t = 2$. A collision will occur at 1:00 pm.
6 Identify any assumptions, and comment on the effects of these assumptions on your solution.	This model assumes there are no underlying currents that would affect the relative velocity of either ferry. It also assumes that the crew of both ferries are willing to continue travelling at a constant velocity despite the obvious dangers of an imminent collision. If these assumptions were incorrect, the ferries would not collide at 1:00 pm.

14 Using vector calculus to describe the motion of a particle

The position vector of a particle at time t is $\boldsymbol{r} = \cos(2t)\hat{\boldsymbol{i}} + \sin(2t)\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$, $t \geq 0$.

(a) Calculate $\dot{\boldsymbol{r}}$ and $\ddot{\boldsymbol{r}}$.

THINKING	WORKING
1 Differentiate each component with respect to t to calculate $\dot{\boldsymbol{r}}$.	$\boldsymbol{r} = \cos(2t)\hat{\boldsymbol{i}} + \sin(2t)\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$ $\dot{\boldsymbol{r}} = \left(\frac{d}{dt}\cos(2t)\right)\hat{\boldsymbol{i}} + \left(\frac{d}{dt}\sin(2t)\right)\hat{\boldsymbol{j}} + \left(\frac{d}{dt}1\right)\hat{\boldsymbol{k}}$ $= -2\sin(2t)\hat{\boldsymbol{i}} + 2\cos(2t)\hat{\boldsymbol{j}}$
2 Differentiate each component with respect to t to calculate $\ddot{\boldsymbol{r}}$.	$\dot{\boldsymbol{r}} = -2\sin(2t)\hat{\boldsymbol{i}} + 2\cos(2t)\hat{\boldsymbol{j}}$ $\ddot{\boldsymbol{r}} = -4\cos(2t)\hat{\boldsymbol{i}} - 4\sin(2t)\hat{\boldsymbol{j}}$

(b) Prove that the speed of the body is constant.

1 Calculate $\lvert\dot{\boldsymbol{r}}\rvert$.	$\dot{\boldsymbol{r}} = -2\sin(2t)\hat{\boldsymbol{i}} + 2\cos(2t)\hat{\boldsymbol{j}}$ $\lvert\dot{\boldsymbol{r}}\rvert = \sqrt{(-2\sin(2t))^2 + (2\cos(2t))^2}$
2 Use the Pythagorean identity to simplify $\lvert\dot{\boldsymbol{r}}\rvert$.	$\lvert\dot{\boldsymbol{r}}\rvert = \sqrt{4\sin^2(2t) + 4\cos^2(2t)}$ $= \sqrt{4(\sin^2(2t) + \cos^2(2t))}$ $= \sqrt{4}$ $= 2$
3 Write the conclusion.	As $\lvert\dot{\boldsymbol{r}}\rvert = 2$, the speed of the body is constant.

(c) Prove that the particle's acceleration is always perpendicular to its direction of motion.

1 Vectors are perpendicular if their dot product is equal to zero. Calculate $\ddot{\boldsymbol{r}} \cdot \dot{\boldsymbol{r}}$.	$\ddot{\boldsymbol{r}} = -4\cos(2t)\hat{\boldsymbol{i}} - 4\sin(2t)\hat{\boldsymbol{j}}$ and $\dot{\boldsymbol{r}} = -2\sin(2t)\hat{\boldsymbol{i}} + 2\cos(2t)\hat{\boldsymbol{j}}$ $\ddot{\boldsymbol{r}} \cdot \dot{\boldsymbol{r}} = \left(-4\cos(2t)\hat{\boldsymbol{i}} - 4\sin(2t)\hat{\boldsymbol{j}}\right) \cdot \left(-2\sin(2t)\hat{\boldsymbol{i}} + 2\cos(2t)\hat{\boldsymbol{j}}\right)$
2 Demonstrate that $\ddot{\boldsymbol{r}} \cdot \dot{\boldsymbol{r}} = 0$.	$\ddot{\boldsymbol{r}} \cdot \dot{\boldsymbol{r}} = (-4\cos(2t))(-2\sin(2t)) + (-4\sin(2t))(2\cos(2t))$ $= 8\sin(2t)\cos(2t) - 8\sin(2t)\cos(2t)$ $= 0$
3 Write the conclusion.	So $\ddot{\boldsymbol{r}} \perp \dot{\boldsymbol{r}}$ since $\ddot{\boldsymbol{r}}, \dot{\boldsymbol{r}} \neq \boldsymbol{0}$. Hence the particle's acceleration is always perpendicular to its direction of motion (the direction of $\dot{\boldsymbol{r}}$).

15 Calculating maximum and minimum speeds of a particle

At time t, a particle has position vector $\boldsymbol{r} = (\cos(t) + \cos(2t))\hat{\boldsymbol{i}} + (\sin(t) + \sin(2t))\hat{\boldsymbol{j}}$, $t \geq 0$.

(a) Determine an expression for the speed of the particle.

THINKING	WORKING
1 Differentiate each component with respect to t to calculate $\dot{\boldsymbol{r}}$.	$\boldsymbol{r} = (\cos(t) + \cos(2t))\hat{\boldsymbol{i}} + (\sin(t) + \sin(2t))\hat{\boldsymbol{j}}$ $\dot{\boldsymbol{r}} = (-\sin(t) - 2\sin(2t))\hat{\boldsymbol{i}} + (\cos(t) + 2\cos(2t))\hat{\boldsymbol{j}}$
2 Recall that speed (a scalar) is the magnitude of the velocity vector. Use $v = \lvert\dot{\boldsymbol{r}}\rvert$ to determine the speed at time t.	$\dot{\boldsymbol{r}} = (-\sin(t) - 2\sin(2t))\hat{\boldsymbol{i}} + (\cos(t) + 2\cos(2t))\hat{\boldsymbol{j}}$ $v = \sqrt{\sin^2(t) + 4\sin(t)\sin(2t) + 4\sin^2(2t) + \cos^2(t) + 4\cos(t)\cos(2t) + 4\cos^2(2t)}$

3 Use the trigonometric identity $\sin^2(\theta)+\cos^2(\theta)=1$ to simplify v.

$$v=\sqrt{\left(\sin^2(t)+\cos^2(t)\right)+4\left(\sin^2(2t)+\cos^2(2t)\right)+4\left(\sin(t)\sin(2t)+\cos(t)\cos(2t)\right)}$$
$$=\sqrt{1+4+4\left(\sin(t)\sin(2t)+\cos(t)\cos(2t)\right)}$$

4 Use the trigonometric identity $\cos(x-y)=\cos(x)\cos(y)+\sin(x)\sin(y)$ to simplify v.

$$v=\sqrt{1+4+4\cos(2t-t)}$$
$$=\sqrt{5+4\cos(t)}$$

5 Write v in terms of t.

$$v=\sqrt{5+4\cos(t)}$$

(b) Determine when the particle first attains its minimum and maximum speed.

1 The minimum speed occurs when $\cos(t)=-1$.

$v=\sqrt{5+4\cos(t)}$

The minimum value of v is 1 and occurs when $\cos(t)=-1$, i.e. when $t=\pi,3\pi,\ldots$

It follows that the minimum speed is 1 and is first attained at $t=\pi$.

2 The maximum speed occurs when $\cos(t)=1$.

The maximum value of v is $\sqrt{9}=3$ and occurs when $\cos(t)=1$, i.e. when $t=0,2\pi,\ldots$

It follows that the maximum speed is 3 and is first attained at $t=0$.

Anti-differentiation of vector functions

Anti-differentiation is the reverse process to differentiation.

Like real (scalar) functions, vector functions can be anti-differentiated.

For anti-differentiating vectors expressed in component form, use the following rule:

If $\frac{d\boldsymbol{r}}{dt}=x\hat{\boldsymbol{i}}+y\hat{\boldsymbol{j}}+z\hat{\boldsymbol{k}}$, then:

$$\boldsymbol{r}=\int\left(x\hat{\boldsymbol{i}}+y\hat{\boldsymbol{j}}+z\hat{\boldsymbol{k}}\right)dt$$
$$=\left(\int x\,dt\right)\hat{\boldsymbol{i}}+\left(\int y\,dt\right)\hat{\boldsymbol{j}}+\left(\int z\,dt\right)\hat{\boldsymbol{k}}+\boldsymbol{c}$$

The result can be verified by differentiation.

$$\frac{d\boldsymbol{r}}{dt}=\frac{d}{dt}\left(\int x\,dt\right)\hat{\boldsymbol{i}}+\frac{d}{dt}\left(\int y\,dt\right)\hat{\boldsymbol{j}}+\frac{d}{dt}\left(\int z\,dt\right)\hat{\boldsymbol{k}}+\frac{d\boldsymbol{c}}{dt}$$
$$=x\hat{\boldsymbol{i}}+y\hat{\boldsymbol{j}}+z\hat{\boldsymbol{k}}+\boldsymbol{0}$$
$$=x\hat{\boldsymbol{i}}+y\hat{\boldsymbol{j}}+z\hat{\boldsymbol{k}}$$

Note that when anti-differentiating a vector function, a *vector* constant ($\boldsymbol{c}$) is introduced.

Extra information, such as an initial condition, is needed to determine the value of the vector constant, $\boldsymbol{c}$ specifically.

16 Determining a position vector from a velocity vector

The velocity of a particle at time t is given by $\dot{\boldsymbol{r}} = \hat{\boldsymbol{i}} + (2t+2)\hat{\boldsymbol{j}}$, $t \geq 0$. Determine its position vector when $t = 1$, if $\boldsymbol{r} = \hat{\boldsymbol{i}}$ when $t = 0$.

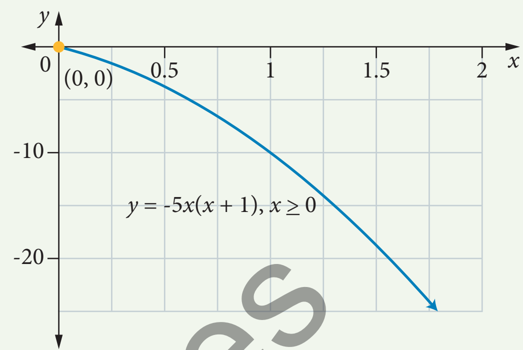
THINKING	WORKING
1 Anti-differentiate each component with respect to t to calculate r.	$\frac{d\boldsymbol{r}}{dt} = \hat{\boldsymbol{i}} + (2t+2)\hat{\boldsymbol{j}}$ $\boldsymbol{r} = \int\left(\hat{\boldsymbol{i}} + (2t+2)\hat{\boldsymbol{j}}\right)dt$ $= t\hat{\boldsymbol{i}} + \left(t^2 + 2t\right)\hat{\boldsymbol{j}} + \boldsymbol{c}$
2 Apply the initial condition to determine $\boldsymbol{c}$.	$\boldsymbol{r}(0) = \hat{\boldsymbol{i}}$ and so $\boldsymbol{c} = \hat{\boldsymbol{i}}$.
3 Write the position vector $\boldsymbol{r}$.	$\boldsymbol{r} = (t+1)\hat{\boldsymbol{i}} + \left(t^2 + 2t\right)\hat{\boldsymbol{j}}$
4 Calculate $\boldsymbol{r}$ when $t = 1$.	$\boldsymbol{r}(1) = 2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$

To calculate the position vector $\boldsymbol{r}$ given $\ddot{\boldsymbol{r}}$, anti-differentiate once to determine $\dot{\boldsymbol{r}}$ and then again to get $\boldsymbol{r}$.

17 Sketching the path of a particle given its acceleration vector

The acceleration of a particle at time t is given by $\ddot{\boldsymbol{r}} = -10\hat{\boldsymbol{j}}$, $t \geq 0$. Sketch the path of the particle if $\dot{\boldsymbol{r}} = \hat{\boldsymbol{i}} - 5\hat{\boldsymbol{j}}$ and $\boldsymbol{r} = 0$ when $t = 0$.

THINKING	WORKING
1 Anti-differentiate each component with respect to t to calculate $\dot{\boldsymbol{r}}$.	$\ddot{\boldsymbol{r}} = -10\hat{\boldsymbol{j}}$ $\dot{\boldsymbol{r}} = -10t\hat{\boldsymbol{j}} + \boldsymbol{c}$
2 Apply the initial condition to determine the constant vector $\boldsymbol{c}$.	At $t = 0$: $\dot{\boldsymbol{r}} = \hat{\boldsymbol{i}} - 5\hat{\boldsymbol{j}}$ Hence $\boldsymbol{c} = \hat{\boldsymbol{i}} - 5\hat{\boldsymbol{j}}$.
3 Anti-differentiate each component with respect to t to calculate $\boldsymbol{r}$.	$\dot{\boldsymbol{r}} = \hat{\boldsymbol{i}} - (10t+5)\hat{\boldsymbol{j}}$ $\boldsymbol{r} = t\hat{\boldsymbol{i}} - \left(5t^2 + 5t\right)\hat{\boldsymbol{j}} + \boldsymbol{d}$
4 Apply the initial condition to determine the constant vector $\boldsymbol{d}$.	At $t = 0$: $\boldsymbol{r} = \boldsymbol{0}$ Hence $\boldsymbol{d} = \boldsymbol{0}$.
5 Write the position vector $\boldsymbol{r}$.	$\boldsymbol{r} = t\hat{\boldsymbol{i}} - \left(5t^2 + 5t\right)\hat{\boldsymbol{j}}$
6 To sketch the path, convert the position vector into Cartesian form. Write and number the parametric equations.	$x = t$ [1] $y = -\left(5t^2 + 5t\right)$ [2]

7	Determine the possible values of x.	If $t \geq 0$ and $x = t$ then $x \geq 0$.
8	Substitute into the other parametric equation to eliminate t.	$y = -\left(5x^2 + 5x\right)$ So $y = -5x(x+1)$.
9	Write the Cartesian equation with domain.	The Cartesian equation is $y = -5x(x+1)$, $x \geq 0$.
10	Sketch the path of the particle with correct domain.	

Technology worked example

Sketching the path of a particle

EXERCISE 3.3 Velocity and acceleration vectors

Worked Example 11

1 Determine an expression for the velocity at time t for a particle that moves so that its position vector at time t is given by:

(a) $\boldsymbol{r} = (t+1)\hat{\boldsymbol{i}} + 2t\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$ (b) $\boldsymbol{r} = t\hat{\boldsymbol{i}} + \left(t^2 - 1\right)\hat{\boldsymbol{j}}$

2 Determine an expression for the acceleration at time t for a particle that moves so that its velocity vector at time t is given by:

(a) $\boldsymbol{v} = \hat{\boldsymbol{i}} - 2t^2\hat{\boldsymbol{j}}$ (b) $\boldsymbol{v} = (15 - 10t)\hat{\boldsymbol{j}}$

3 Determine $\dot{\boldsymbol{r}}$ and $\ddot{\boldsymbol{r}}$ for each of the following equations.

(a) $\boldsymbol{r} = 4t\hat{\boldsymbol{i}} - \left(t^2 - 2\right)\hat{\boldsymbol{j}} + 3t^2\hat{\boldsymbol{k}}$ (b) $\boldsymbol{r} = 26t\hat{\boldsymbol{i}} + 5\left(3t - t^2\right)\hat{\boldsymbol{j}}$

(c) $\boldsymbol{r} = 4\cos(2t)\hat{\boldsymbol{i}} + 3\sin(2t)\hat{\boldsymbol{j}}$ (d) $\boldsymbol{r} = e^t\hat{\boldsymbol{i}} + \left(e^{-2t} + 1\right)\hat{\boldsymbol{j}} - t^3\hat{\boldsymbol{k}}$

4 A particle moves so that its position vector at time t is given by $\boldsymbol{r} = 2\cos(t)\hat{\boldsymbol{i}} + 3\sin(t)\hat{\boldsymbol{j}} - 4t\hat{\boldsymbol{k}}$. Calculate the initial speed of the particle.

5 A particle moves so that at time $t \geq 0$, its position vector is given by $\boldsymbol{r} = \left(t^3 - t\right)\hat{\boldsymbol{i}} + (2t-1)^3\hat{\boldsymbol{j}}$. At $t = 1$, the magnitude of the particle's acceleration is:

A $2\hat{\boldsymbol{i}} + 6\hat{\boldsymbol{j}}$ B $2\sqrt{10}$ C $6\hat{\boldsymbol{i}} + 24\hat{\boldsymbol{j}}$ D $6\sqrt{17}$

6 Calculate the position vector $\boldsymbol{r}$ if: 16

(a) $\dot{\boldsymbol{r}} = 4t\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}}$, $\boldsymbol{r} = \hat{\boldsymbol{j}}$ at $t = 0$.

(b) $\dot{\boldsymbol{r}} = -2\sin(2t)\hat{\boldsymbol{i}} + 2\cos(2t)\hat{\boldsymbol{j}}$, $\boldsymbol{r} = \hat{\boldsymbol{i}} + \hat{\boldsymbol{k}}$ at $t = 0$.

(c) $\ddot{\boldsymbol{r}} = -10\hat{\boldsymbol{j}}$, $\boldsymbol{r} = \boldsymbol{0}$ and $\dot{\boldsymbol{r}} = 20\hat{\boldsymbol{i}} + 20\hat{\boldsymbol{j}}$ at $t = 0$.

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7 Two snooker balls, the white ball and the black ball, simultaneously rebound slowly off adjacent cushions of a snooker table. Immediately after rebounding, their position and velocity vectors relative to a corner of the table are: 13

$\boldsymbol{r}_W = 0.5\hat{\boldsymbol{i}}$ $\quad \boldsymbol{v}_W = \hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}$

$\boldsymbol{r}_B = 1.5\hat{\boldsymbol{j}}$ $\quad \boldsymbol{v}_B = 2\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}}$

where distance is measured in metres and time in seconds.

(a) Prove that the paths of the two balls cross at right angles.

(b) Calculate how long after rebounding the balls collide.

(c) Identify any assumptions made in parts (a) and (b) and discuss the effects of these assumptions on your answer.

8 The position vector of a particle at time t is given by $\boldsymbol{r} = \sqrt{2}\cos(nt)\hat{\boldsymbol{i}} + \sin(nt)\hat{\boldsymbol{j}} + \sin(nt)\hat{\boldsymbol{k}}$, where n is a positive constant. 14

(a) Prove that the particle moves with constant speed.

(b) Prove that its acceleration is always perpendicular to its velocity.

9 A particle moves so that its position vector at time t is given by $\boldsymbol{r} = \left(e^{-t}\sin(t)\right)\hat{\boldsymbol{i}} + \left(e^{-t}\cos(t)\right)\hat{\boldsymbol{j}}$, $t \geq 0$.

(a) Determine an expression for the particle's velocity.

(b) Determine an expression for the particle's speed.

(c) Determine an expression for the particle's acceleration.

10 At time t, a particle has position vector $\boldsymbol{r} = \left(3\sin(t) + \sin(2t)\right)\hat{\boldsymbol{i}} + \left(3\cos(t) - \cos(2t)\right)\hat{\boldsymbol{j}} + t\hat{\boldsymbol{k}}$, $t \geq 0$. 15

(a) Calculate the maximum and minimum speeds of the particle.

(b) Calculate the magnitude of its acceleration at $t = \dfrac{\pi}{2}$.

11 A particle moves so that at time $t > 0$, its position vector is given by $\boldsymbol{r} = \ln\left(t^2 + 2t\right)\hat{\boldsymbol{i}} + 2t^2\hat{\boldsymbol{j}}$. Calculate the particle's velocity vector at $t = 2$.

12 The acceleration of a particle moving in a plane is constant and is given by $\ddot{\boldsymbol{r}} = 18\hat{\boldsymbol{i}}$. At $t = 0$, $\dot{\boldsymbol{r}} = 6\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$ and $\boldsymbol{r} = \boldsymbol{0}$. Calculate the velocity and the position of the particle at $t = 1$. 17

13 The acceleration of a particle at time t is given by $\ddot{\boldsymbol{r}} = e^{t}\hat{\boldsymbol{i}} + e^{-t}\hat{\boldsymbol{j}} + 6t\hat{\boldsymbol{k}}$. Calculate the particle's velocity and position at time t if $\dot{\boldsymbol{r}} = \hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$ and $\boldsymbol{r} = 2\hat{\boldsymbol{j}}$ at $t = 0$.

14 The acceleration at time t seconds of a particle moving in a straight line is $\ddot{\boldsymbol{x}} = (4 - 6t)\hat{\boldsymbol{i}}$ m/s^2. The particle starts its motion 16 metres to the right of a fixed point O and moves away from O at a speed of 4 m/s.

(a) Determine the position where the particle is instantaneously at rest, relative to O.

(b) Determine when the particle passes through O.

15 A body moves from rest at the origin so that after t seconds its acceleration is given by $\ddot{\boldsymbol{r}} = 10\hat{\boldsymbol{i}} + e^{-0.1t}\hat{\boldsymbol{j}}\,\text{m/s}^2$.

Worked Example 12

(a) Calculate its velocity after t seconds.

(b) Calculate its position vector at $t = 10$.

16 A particle moves so that its position vector at time t is $\boldsymbol{r} = t^2\hat{\boldsymbol{i}} + t\hat{\boldsymbol{j}},\ t \geq 0$.

(a) Determine an expression for the velocity at time t.

(b) Calculate the speed and direction of the particle at $t = 2$.

(c) Determine an expression for the acceleration at time t.

(d) Resolve the acceleration vector at $t = 1$ into tangential and normal components.

(e) At what angle to its direction of motion is the particle accelerating at $t = 0$ and $t = 1$?

17 A child is sitting still in some long grass watching a bee. The bee flies at constant speed in a straight line from its beehive to a flower and reaches the flower 3 seconds later. The position vector of the beehive relative to the child is $10\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + 6\hat{\boldsymbol{k}}$ and the position vector of the flower relative to the child is $7\hat{\boldsymbol{i}} + 8\hat{\boldsymbol{j}}$, where all distances are measured in metres.

(a) Calculate the speed of the bee.

(b) Calculate a unit vector in the direction of motion of the bee.

(c) Calculate the velocity of the bee.

(d) Calculate the closest distance that the bee comes to the child. Express your answer correct to one decimal place.

18 The velocity of a golf ball, t seconds, after being hit from a tee, is given by $\dot{\boldsymbol{r}} = 10\hat{\boldsymbol{i}} + (20 - 10t)\hat{\boldsymbol{j}}$ m/s where $\hat{\boldsymbol{i}}$ is a unit vector in the horizontal direction of the green and $\hat{\boldsymbol{j}}$ is a unit vector vertically upward.

(a) Calculate the initial speed and the angle of projection of the ball. Give your answers correct to one decimal place.

(b) Determine when the ball's velocity is horizontal.

(c) Taking the tee as the origin, determine an expression for the position vector $\boldsymbol{r}$ of the ball at time t seconds.

(d) Calculate the maximum height reached by the ball. Identify any assumptions and comment on the effect of these assumptions on your solution.

19 At time t, a particle has velocity $\boldsymbol{v} = 2\cos(t)\hat{\boldsymbol{i}} - 4\sin(t)\cos(t)\hat{\boldsymbol{j}},\ t \geq 0$. At time $t = 0$, $\boldsymbol{r} = 3\hat{\boldsymbol{j}}$.

(a) Calculate the displacement of the particle at any time t.

(b) Calculate the displacement of the particle when it first comes to rest.

(c) Determine the Cartesian equation of the path and sketch the path of the particle.

(d) Determine the maximum speed attained by the particle.